Math 4300

Homework #4

Solutions

$\boxed{1}$ $A = (-1,-2)$, $B = (2,1)$, $C = (0,-1)$

(a) They don't lie on a vertical line.
What about a line $L_{m,b}$?
Plug them into $y = mx + b$ to get:

$$\boxed{\begin{aligned} -2 &= -m + b \\ 1 &= 2m + b \\ -1 &= b \end{aligned}} \quad \begin{aligned} &① \leftarrow A \text{ plugged in} \\ &② \leftarrow B \text{ plugged in} \\ &③ \leftarrow C \text{ plugged in} \end{aligned}$$

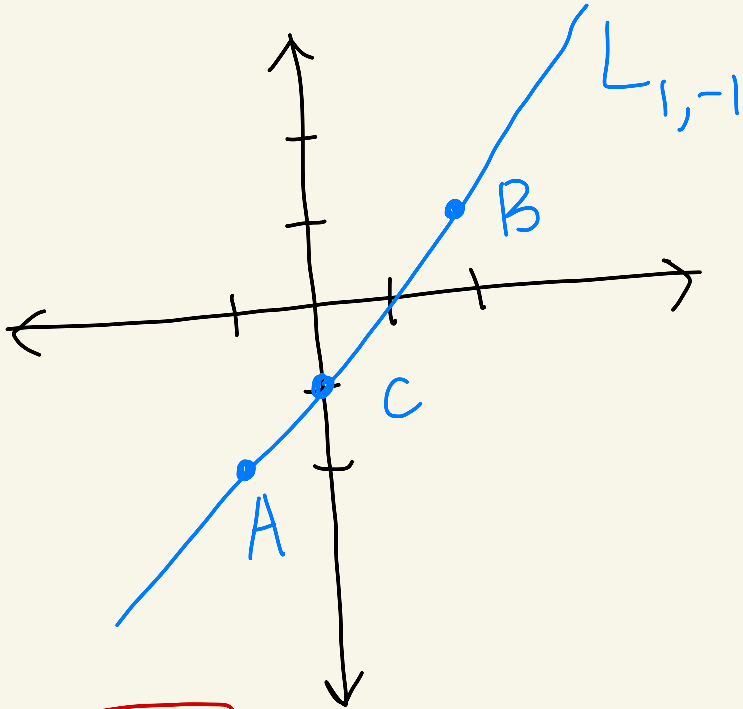
$\Downarrow$

$b = -1$ gives then $m = 1$ in both ① and ②.
Let's verify that all three points satisfy the equation $y = x - 1$.

We have:

$$-2 = -1 - 1 \checkmark$$
$$1 = 2 - 1 \checkmark$$
$$-1 = 0 - 1 \checkmark$$

These three points all lie on $L_{m,b} = L_{1,-1}$.
So they are collinear.



$\boxed{\text{Method 1 - by def}}$

(b) In the picture on the previous page we can guess that $A-C-B$ is true.

Let's check:

(i) we have three distinct points ✓

(ii) A, B, C are collinear ✓

(iii) $d_E(A,C) + d_E(C,B)$

$$= \sqrt{(-1-0)^2 + (-2+1)^2} + \sqrt{(0-2)^2 + (-1-1)^2}$$

$$= \sqrt{2} + \sqrt{8} = \sqrt{2}(1+2) = 3\sqrt{2}$$

$\boxed{\begin{array}{l} A = (-1,-2) \\ B = (2,1) \\ C = (0,-1) \end{array}}$

And, $d_E(A,B) = \sqrt{(-1-2)^2 + (-2-1)^2}$

$$= \sqrt{9+9}$$

$$= \sqrt{18} = 3\sqrt{2}$$

So, $d_E(A,C) + d_E(C,B) = d_E(A,B)$.

By conditions (i), (ii), (iii) we have that A-C-B is true.

By a later problem of this HW we cannot have also A-B-C or B-A-C. Thus, only A-C-B is true.

$\boxed{\text{method 2}}$

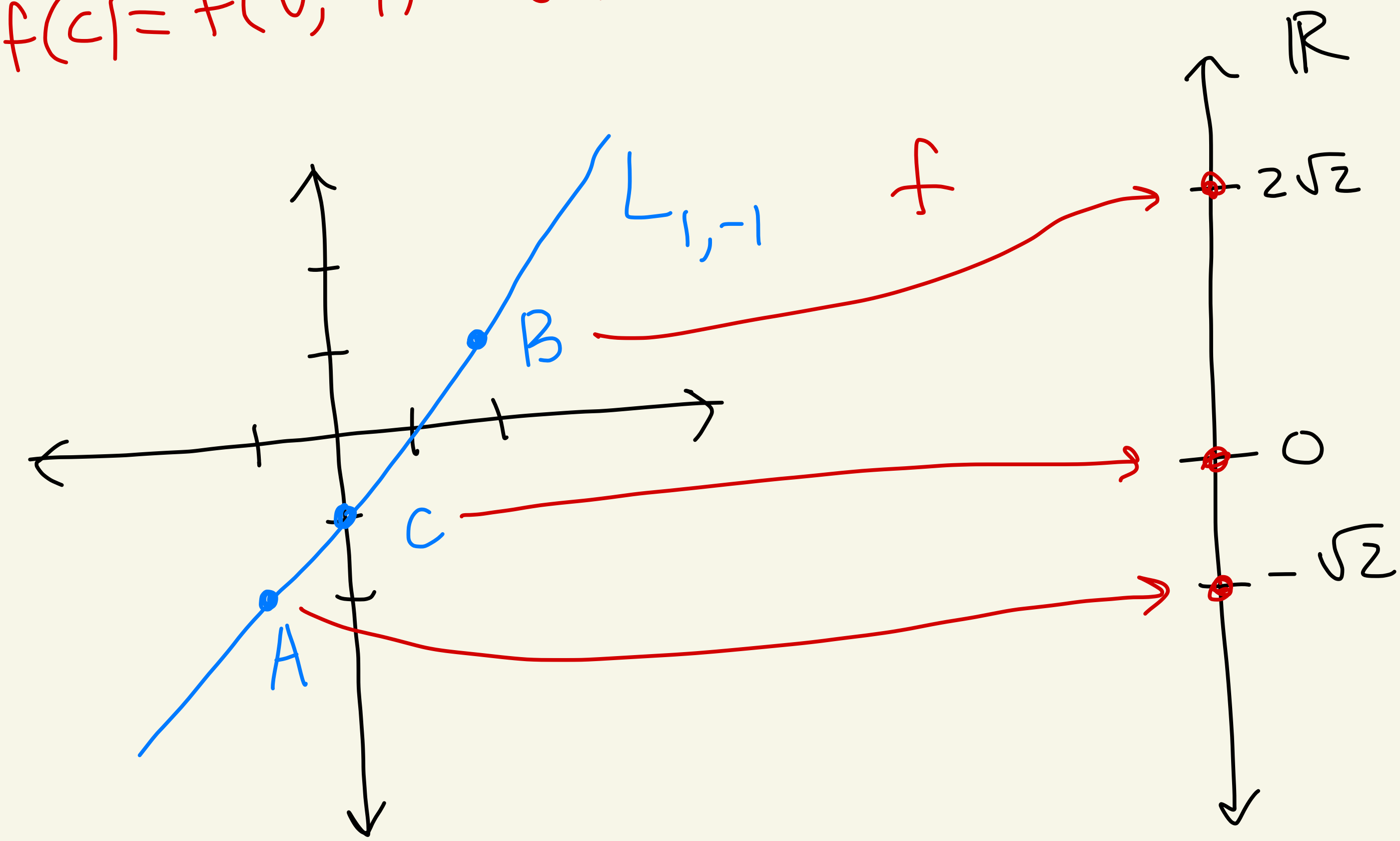
Note: There is another way to check condition (iii) above. By using the standard ruler!

The standard ruler on $L_{1,-1}$.

The standard ruler is $f: L_{1,-1} \to \mathbb{R}$ where $f(x,y) = x\sqrt{1+1^2} = \sqrt{2}\,x$

Apply f to A, B, C to get

$$f(A) = f(-1,-2) = -\sqrt{2}$$
$$f(B) = f(2,1) = 2\sqrt{2}$$
$$f(C) = f(0,-1) = 0\sqrt{2} = 0$$



Since $f(A) < f(C) < f(B)$ we have $A - C - B$ by a theorem from class.

(c) Since $A - C - B$, then by a theorem in class we also have $B - C - A$.

$\boxed{2}$ $A=(1,2)$, $B=(3,4)$, $C=(4,\sqrt{19})$

(a) They aren't on a vertical line.
Let's see if they are on some $_c L_r$.
Plug A, B into $(x-c)^2+y^2=r^2$ to get:
(we only need to first find $\overleftrightarrow{AB}$ and then check if C lies on it also)

$$\boxed{\begin{aligned}(1-c)^2+2^2&=r^2 \quad (1) \leftarrow \text{A plugged in}\\ (3-c)^2+4^2&=r^2 \quad (2) \leftarrow \text{B plugged in}\end{aligned}}$$

$\Downarrow$

$$\boxed{\begin{aligned}c^2-2c+5&=r^2 \quad (1)\\ c^2-6c+25&=r^2 \quad (2)\end{aligned}}$$

(1) − (2) gives $4c-20=0$. So, $c=5$.
Plugging $c=5$ into (1) gives $r=\sqrt{20}$

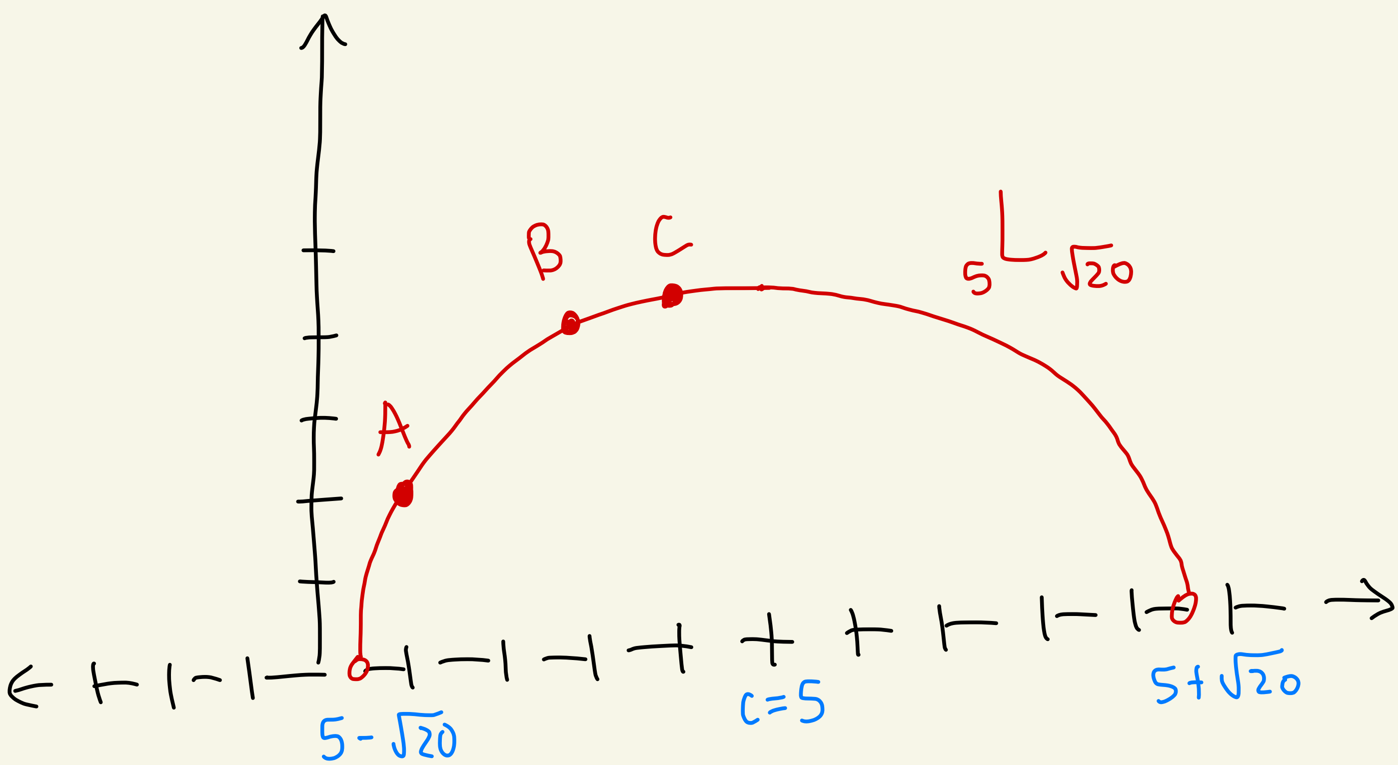
Now let's see if all three points satisfy $(x-5)^2+y^2=20$

$$(1-5)^2 + 2^2 = 20 \checkmark$$
$$(3-5)^2 + 4^2 = 20 \checkmark$$
$$(4-5)^2 + \sqrt{19}^2 = 20 \checkmark$$

$A = (1, 2)$
$B = (3, 4)$
$C = (4, \sqrt{19})$

So, A, B, C all lie on ${}_5L_{\sqrt{20}}$

Note $\sqrt{20} \approx 4.47$ and $\sqrt{19} \approx 4.36$



$\boxed{\text{method 1 - by def}}$

(b) From the picture it looks like $A-B-C$. Let's verify.

(i) We have three distinct points

(ii) A, B, C are collinear

(iii) $d_H(A,B) + d_H(B,C)$

$A = (1,2)$
$B = (3,4)$
$C = (4,\sqrt{19})$

$$= \left|\ln\left(\frac{(1-5+\sqrt{20})/2}{(3-5+\sqrt{20})/4}\right)\right| + \left|\ln\left(\frac{(3-5+\sqrt{20})/4}{(4-5+\sqrt{20})/\sqrt{19}}\right)\right|$$

recall: If
$P = (x_1, y_1)$
$Q = (x_2, y_2)$
are on ${}_cL_r$

$$d(P,Q) = \left|\ln\left(\frac{(x_1-c+r)/y_1}{(x_2-c+r)/y_2}\right)\right|$$

$$= \left|\underbrace{\ln\left(\frac{(-4+\sqrt{20})/2}{(-2+\sqrt{20})/4}\right)}_{\approx -2.34 < 0}\right| + \left|\underbrace{\ln\left(\frac{(-2+\sqrt{20})/4}{(-1+\sqrt{20})/\sqrt{19}}\right)}_{\approx -0.60 < 0}\right|$$

$$= -\ln\left(\frac{(-4+\sqrt{20})/2}{(-2+\sqrt{20})/4}\right) - \ln\left(\frac{(-2+\sqrt{20})/4}{(-1+\sqrt{20})/\sqrt{19}}\right)$$

the negative signs are coming from dropping the abs value since the # inside the abs value is negative

$-\ln(c) = \ln\left(\frac{1}{c}\right)$

$$= \ln\left(\frac{(-2+\sqrt{20})/4}{(-4+\sqrt{20})/2}\right) + \ln\left(\frac{(-1+\sqrt{20})/\sqrt{19}}{(-2+\sqrt{20})/4}\right)$$

$\ln(A) + \ln(B) = \ln(AB)$

$$= \underbrace{\ln\left(\frac{(-1+\sqrt{20})/\sqrt{19}}{(-4+\sqrt{20})/2}\right)}_{\approx 0.963 > 0}$$

$$= \left|\ln\left(\frac{(-1+\sqrt{20})/\sqrt{19}}{(-4+\sqrt{20})/2}\right)\right|$$

$$= \left|\ln\left(\frac{(4-5+\sqrt{20})/19}{(1-5+\sqrt{20})/2}\right)\right|$$

$$= d_H(C,A)$$

property of distance functions

$$= d_H(A,C)$$

Thus, from (i), (ii), (iii) we see that $A - B - C$. From a hw problem is this topic we can't also have $A - C - B$ or $B - A - C$.

$\boxed{\text{Method 2}}$ - There is another way to check condition (iii). Use the standard ruler! The standard ruler on ${}_5L_{\sqrt{20}}$ is given by $f: {}_5L_{\sqrt{20}} \longrightarrow \mathbb{R}$ where $f(x,y) = \ln\left(\frac{x-5+\sqrt{20}}{y}\right)$

We have

$$f(A) = f(1,2) = \ln\left(\frac{1-5+\sqrt{20}}{2}\right) \approx -1.44$$

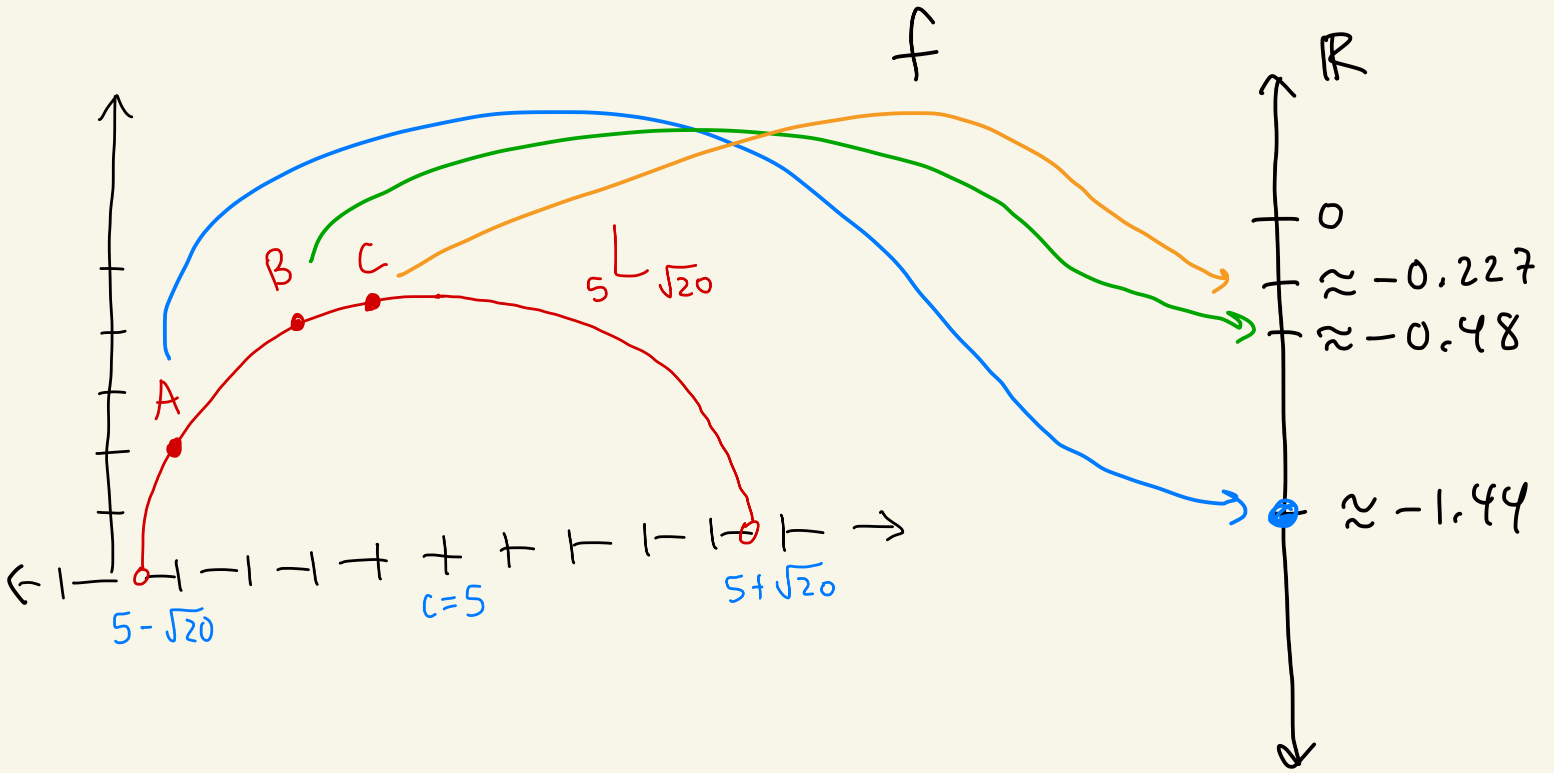
$$f(B) = f(3,4) = \ln\left(\frac{3-5+\sqrt{20}}{4}\right) \approx -0.48$$

$$f(C) = f(4,\sqrt{19}) = \ln\left(\frac{4-5+\sqrt{20}}{\sqrt{19}}\right) \approx -0.227$$

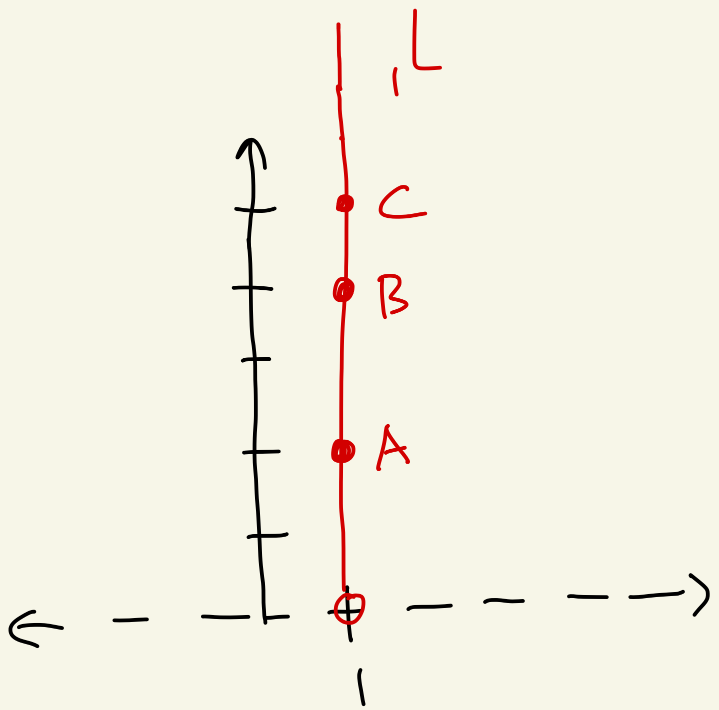
Since $f(C) < f(B) < f(A)$

We know $C - B - A$ or equivalently from a theorem from class

We have A – B – C.



(3) (a) All three points $A=(1,2)$, $B=(1,4)$, and $C=(1,5)$ lie on ${}_1L$ and so they are collinear.



(b) Let's check if $A-B-C$, $A-C-B$, or $B-A-C$.

You can use the def way of doing this like in the previous problems but this time let's use the easier standard ruler method (method 2 in the previous problems)

(i) A, B, C are distinct points ✓

(ii) A, B, C are collinear ✓

(iii) the standard ruler on ${}_1L$ is given by $f(1,y) = \ln(y)$

We have

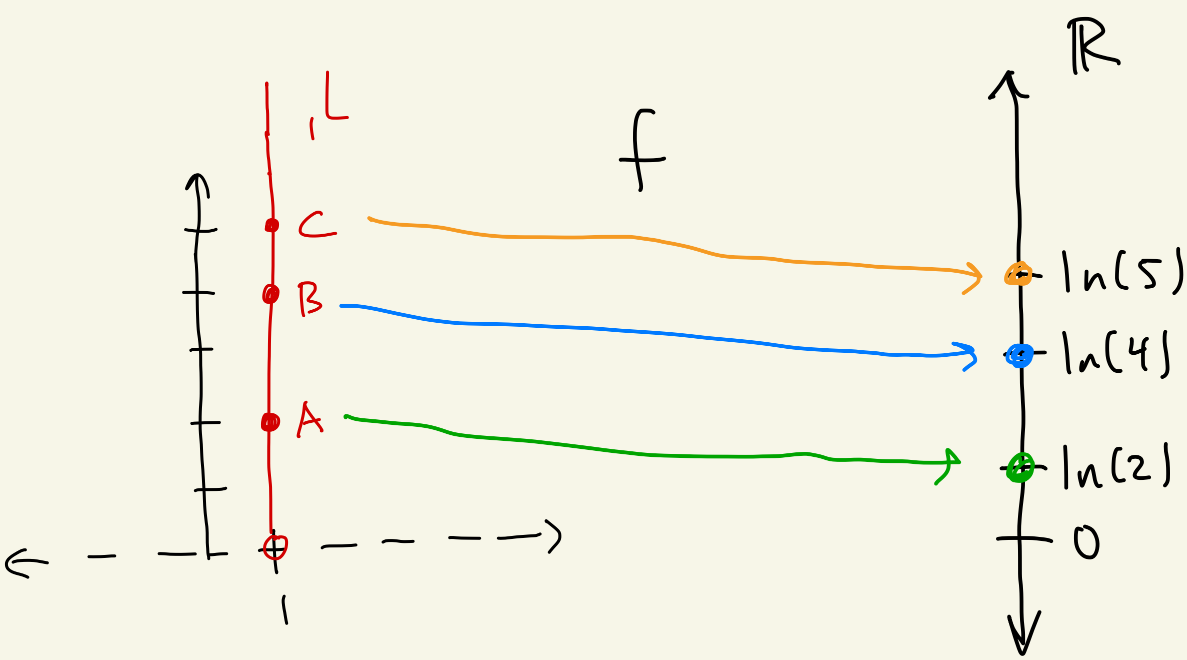
$f(A) = f(1,2) = \ln(2) \approx 0.693$

$f(B) = f(1,4) = \ln(4) \approx 1.386$

$f(C) = f(1,5) = \ln(5) \approx 1.609$

Since $f(A) < f(B) < f(C)$

we know $A - B - C$.



(4) Let A, B be points with $A \neq B$ in a metric geometry.

Let $C \in \overleftrightarrow{AB}$.

Let $f: \overleftrightarrow{AB} \to \mathbb{R}$ be a ruler for $\overleftrightarrow{AB}$.

Since $A \neq B$ and f is a bijection we know that $f(A) \neq f(B)$.

Then there are several cases to consider:

(i) $f(A) < f(B) < f(C)$

(ii) $f(A) < f(C) < f(B)$

(iii) $f(A) = f(C)$

(iv) $f(B) < f(A) < f(C)$

(v) $f(B) < f(C) < f(A)$

(vi) $f(B) = f(C)$

(vii) $f(C) < f(A) < f(B)$

(viii) $f(C) < f(B) < f(A)$

If (i) is true then $A-B-C$.

If (ii) is true then $A-C-B$

If (iii) is true then $A=C$. since f is 1-1

If (iv) is true then $B-A-C$ which implies that $C-A-B$.

from a thm in class: If $X-Y-Z$ then $Z-Y-X$

If (v) is true then $B-C-A$ which implies that $A-C-B$.

If (vi) is true then $B=C$ since f is 1-1

If (vii) is true then $C-A-B$.

If (viii) is true then $C-B-A$, which implies that $A-B-C$.

Thus summarizing either:

$C-A-B$, or $C=A$, or $A-C-B$,

or $C=B$, or $A-B-C$.

And no two of these can be true at the same time (by the above cases argument) $\boxtimes$

5. Let ℓ be a line and A, B, C be distinct points on ℓ in a metric geometry.

So, $A \neq B$, $A \neq C$, and $B \neq C$.

We know $\ell = \overleftrightarrow{AB}$ since there is a unique line through any two distinct points.

By the previous HW problem we know that there are only three possible outcomes (since $C \neq A$ and $C \neq B$)

These are that either $C-A-B$, or $A-C-B$, or $A-B-C$.

Since $C-A-B$ implies $B-A-C$, this gives that the only three possibilities are $B-A-C$, or $A-C-B$, or $A-B-C$. $\blacksquare$

6. Suppose that $A-B-C$ and $B-C-D$ in some metric geometry.

Since $A-B-C$ we know A, B, C are distinct and collinear and all lie on $\overleftrightarrow{BC}$.

Since $B-C-D$ we know that B, C, D are all distinct and collinear and all lie on $\overleftrightarrow{BC}$.

Let $f : \overleftrightarrow{BC} \to \mathbb{R}$ be a ruler.

Since $A-B-C$ we know either

(i) $f(A) < f(B) < f(C)$

or (ii) $f(C) < f(B) < f(A)$.

Since $B-C-D$ we know either

(iii) $f(B) < f(C) < f(D)$

or (iv) $f(D) < f(C) < f(B)$.

$\boxed{\text{case 1}}$:

Suppose (i) $f(A) < f(B) < f(C)$ is true.

Then, we can't have (iv) since then $f(C) < f(B)$.

So we must have (iii), that is $f(B) < f(C) < f(D)$.

Thus, $f(A) < f(B) < f(C) < f(D)$.

So, $A - B - D$ and $A - C - D$.

$\boxed{\text{Case 2}}$:

Suppose (ii) $f(C) < f(B) < f(A)$ is true.

Then, we can't have (iii) since then $f(B) < f(C)$

So we must have (iv), that is $f(D) < f(C) < f(B)$.

Thus, $f(D) < f(C) < f(B) < f(A)$.

Hence, $D - B - A$ and $D - C - A$.

So, $A - B - D$ and $A - C - D$. $\blacksquare$

7. Suppose $A-C-D$ and $A-C-B$ is some metric geometry. And that $D \neq B$.

Then, $A, B, C, D \in \overleftrightarrow{AC}$.

Let $\ell = \overleftrightarrow{AC}$ and $f: \ell \to \mathbb{R}$ be a ruler.

Since $A-C-D$ we get either

$$\text{(i) } f(A) < f(C) < f(D)$$

$$\text{or (ii) } f(D) < f(C) < f(A)$$

Case (i): Suppose $f(A) < f(C) < f(D)$.

Since $A-C-B$ we have either

$$f(A) < f(C) < f(B) \quad (*)$$

$$\text{or} \quad f(B) < f(C) < f(A) \quad (**)$$

But $(**)$ can't happen since we are assuming that $f(A) < f(C)$.

Thus we must have $(*)$.

Combining the case (i) conditions with $(*)$

we get either

$$f(A) < f(C) < f(D) < f(B)$$

or $f(A) < f(C) < f(B) < f(D)$.

In the first inequality we get $A-D-B$.

In the second we get $A-B-D$.

So either $A-D-B$ or $A-B-D$.

<u>Case (ii)</u>: Suppose $f(D) < f(C) < f(A)$.

Since $A-C-B$ we have either

$$f(A) < f(C) < f(B) \quad (***)$$

or $f(B) < f(C) < f(A) \quad (****)$

We see that $(***)$ can't happen because we are assuming that $f(C) < f(A)$.

Thus we must have $(****)$

Combining case (ii) with $(****)$ we get that either

$$f(D) < f(B) < f(C) < f(A)$$

or $$f(B) < f(D) < f(C) < f(A).$$

Thus either $D - B - A$ or $B - D - A$.

So either $A - B - D$ or $A - D - B$.

$\blacksquare$

8 Suppose $A-D-C$ and $A-C-B$.

Then A, B, C, D all lie on $\ell = \overleftrightarrow{AC}$.

Let $f : \ell \to \mathbb{R}$ be a ruler.

Since $A-D-C$ we have either

$$(i) \quad f(A) < f(D) < f(C)$$

$$\text{or } (ii) \quad f(C) < f(D) < f(A)$$

$\boxed{\text{case (i):}}$ Suppose $f(A) < f(D) < f(C)$.

Since $A-C-B$ we have either

$$f(A) < f(C) < f(B) \qquad (*)$$

$$\text{or} \quad f(B) < f(C) < f(A) \qquad (**)$$

We can't have $(**)$ since we are assuming in this case that $f(A) < f(C)$.

Thus we have $(*)$.

We get that case (i) and (*) give

that $f(A) < f(D) < f(C) < f(B)$.

Thus, $A - D - B$.

Case (ii): Suppose $f(C) < f(D) < f(A)$.

Since $A - C - B$ we have either

$$f(A) < f(C) < f(B) \quad (***)$$

or

$$f(B) < f(C) < f(A) \quad (****)$$

We can't have (***) since we are assuming in this case that $f(C) < f(A)$.

Thus we have (****).

We get that case (ii) and (****) give

that $f(B) < f(C) < f(D) < f(A)$.

Thus, $B - D - A$.

So, $A - D - B$.

In either case we get $A - D - B$. $\blacksquare$

9 Suppose that $A-Q-B$, $A-P-B$, and $P-C-Q$. Let $\ell = \overleftrightarrow{AB}$.

Since $A-Q-B$ and $A-P-B$ we know all of A, B, P, Q lie on ℓ.

Let $f: \ell \to \mathbb{R}$ be a ruler for ℓ.

Since $A-Q-B$ we get two cases: either $f(A) < f(Q) < f(B)$ <u>or</u> $f(B) < f(Q) < f(A)$.

I'll prove this problem for when $f(A) < f(Q) < f(B)$, you try the other case.

$\boxed{\text{Suppose } f(A) < f(Q) < f(B).}$ $(*)$

Since $A-P-B$ we have two cases.

<u>case 1:</u> Suppose $\boxed{f(A) < f(P) < f(B)}$ (1)

Since $P-C-Q$ we have either $\underbrace{f(P) < f(C) < f(Q)}_{(i)}$ or $\underbrace{f(Q) < f(C) < f(P)}_{(ii)}$.

If (i), then

$$f(A) \underset{(1)}{<} f(P) \underset{(i)}{<} f(C) \underset{(i)}{<} f(Q) \underset{(*)}{<} f(B)$$

So, $A-C-B$.

If (ii), then

$$f(A) \underset{(*)}{<} f(Q) \underset{(ii)}{<} f(C) \underset{(ii)}{<} f(P) \underset{(1)}{<} f(B)$$

So, $A - C - B$.

<u>Case 2:</u> Suppose $\boxed{f(B) < f(P) < f(A)}$ (2)

Since $P - C - Q$ we have either

$$\underbrace{f(P) < f(C) < f(Q)}_{(i)} \quad \text{or} \quad \underbrace{f(Q) < f(C) < f(P)}_{(ii)}.$$

If (i), then

$$f(B) \underset{(2)}{<} f(P) \underset{(i)}{<} f(C) \underset{(i)}{<} f(Q) \underset{(*)}{<} f(B)$$

This is a contradiction, so this case can't happen.

If (ii), then

$$f(A) \underset{(*)}{<} f(Q) \underset{(*)}{<} f(B) \underset{(2)}{<} f(P) \underset{(2)}{<} f(A)$$

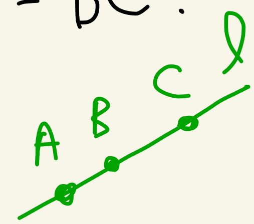
This is a contradiction, so this case can't happen.

$\blacksquare$

(10) Let $A, B, C \in \mathbb{R}^2$ be distinct points.

$\boxed{(\Rightarrow)}$ Suppose $A - B - C$.

Then, A, B, C all lie on $l = \overleftrightarrow{AB} = \overleftrightarrow{AC} = \overleftrightarrow{BC}$.



Since $B \in l = \overleftrightarrow{AC}$ and $\overleftrightarrow{AC} = L_{AC}$ we know that $\boxed{B = A + t(C - A)}$ where $t \in \mathbb{R}$.

Goal:

We must show that $0 < t < 1$

If we can do this then we have proven the $(\Rightarrow)$ direction of this proof

Let $A = (x_a, y_a)$, $B = (x_b, y_b)$ and $C = (x_c, y_c)$.

Then, $B = A + t(C - A)$ becomes

$$\underbrace{(x_b, y_b)}_{B} = \underbrace{(x_a + t x_c - t x_a,\ y_a + t y_c - t y_a)}_{A + t(C - A)} \quad (*)$$

We now break the proof into two cases: if ℓ is a vertical line and if ℓ is a non-vertical line.

Case 1: Suppose $\ell = L_d$ is a vertical line. Let $f : \ell \longrightarrow \mathbb{R}$ be the standard ruler given by $f(x,y) = y$.

Apply the ruler f to $(*)$ above to get

$$y_b = y_a + t y_c - t y_a.$$

So,

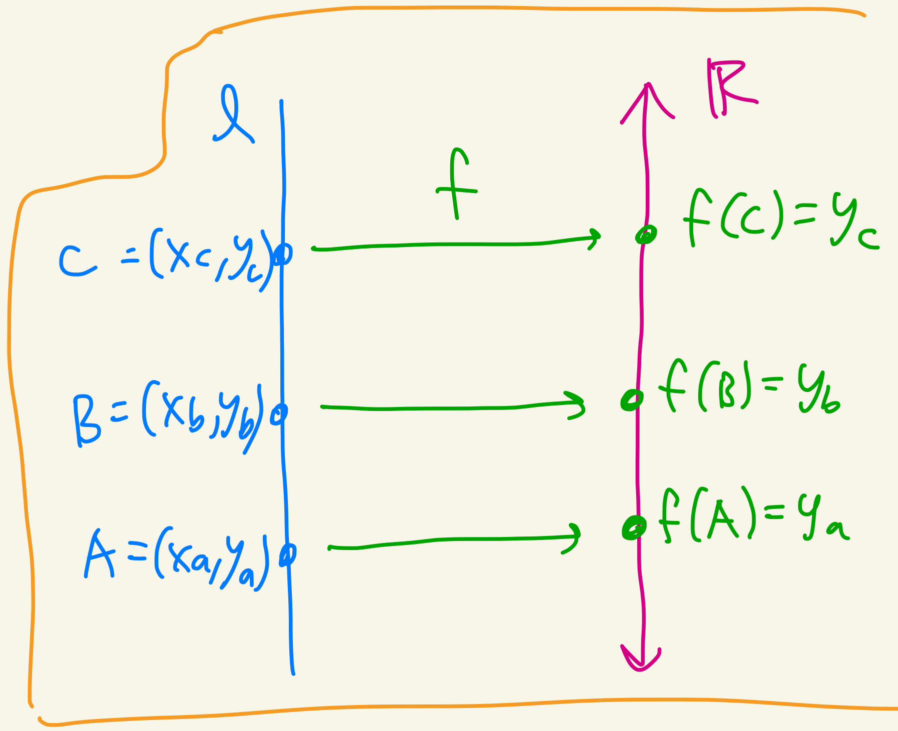
$$\boxed{y_b - y_a = t(y_c - y_a)} \quad (**)$$

Since $A-B-C$ we know that either

$$\text{(i)} \quad f(A) < f(B) < f(C)$$

or $$\text{(ii)} \quad f(C) < f(B) < f(A)$$

vertical line pic



Thus either

$$\text{(i)} \quad y_a < y_b < y_c$$

or $$\text{(ii)} \quad y_c < y_b < y_a$$

<u>case (i)</u>: Suppose we have (i).

That is, suppose $y_a < y_b < y_c$.

Then, $0 < y_b - y_a$ and $0 < y_c - y_a$.

In (**) we have $\underbrace{y_b - y_a}_{>0} = t\,\underbrace{(y_c - y_a)}_{>0}$

So we must have $t > 0$.

Why can't $t \geq 1$?

Suppose $t \geq 1$.

Then from $(*)$ we get

$$y_b - y_a = t(y_c - y_a) \geq y_c - y_a$$

But then $y_b \geq y_c$.

This contradicts $y_b < y_c$ from (i).

Therefore $0 < t < 1$ and we are done with this case

case (ii): Suppose (ii), that is, $y_c < y_b < y_a$.

Then, $0 < y_a - y_c$ and $0 < y_a - y_b$.

So, $y_c - y_a < 0$ and $y_b - y_a < 0$.

Thus we have from $(*)$ $\underbrace{(y_b - y_a)}_{<0} = t\underbrace{(y_c - y_a)}_{<0}$

So, $t > 0$.

Why can't we have $t \geq 1$?

Suppose $t \geq 1$.

Then $\frac{1}{t} \le 1$ and $\frac{1}{t}(y_b - y_a) = (y_c - y_a)$

$\times(-1)$

Thus, $\frac{1}{t}(y_a - y_b) = y_a - y_c$.

Then, $y_a - y_c = \underbrace{\frac{1}{t}}_{\le 1}(y_a - y_b) \le y_a - y_b$.

So, $-y_c \le -y_b$

Then $y_c \ge y_b$.

But this contradicts $y_c < y_b$ from (ii)

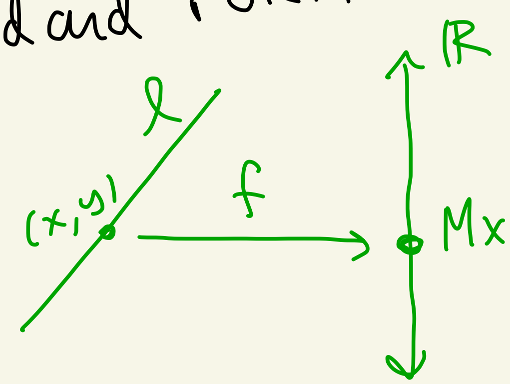
Therefore, $t \ge 1$ can't be true
and thus $0 < t < 1$.

This completes the proof of case 1.

$\boxed{\text{Case 2:}}$ Suppose $\ell = L_{m,c}$ is a non-vertical line.

Let $f : \ell \to \mathbb{R}$ be the standard ruler.

Then, $f(x,y) = x\sqrt{1+m^2} = xM$
where $M = \sqrt{1+m^2}$



Recall that (*) says that

$(x_b, y_b) = (x_a + tx_c - tx_a, y_a + ty_c - ty_a)$

Applying f to the above eqn gives:

$$Mx_b = M(x_a + t x_c - t x_a)$$

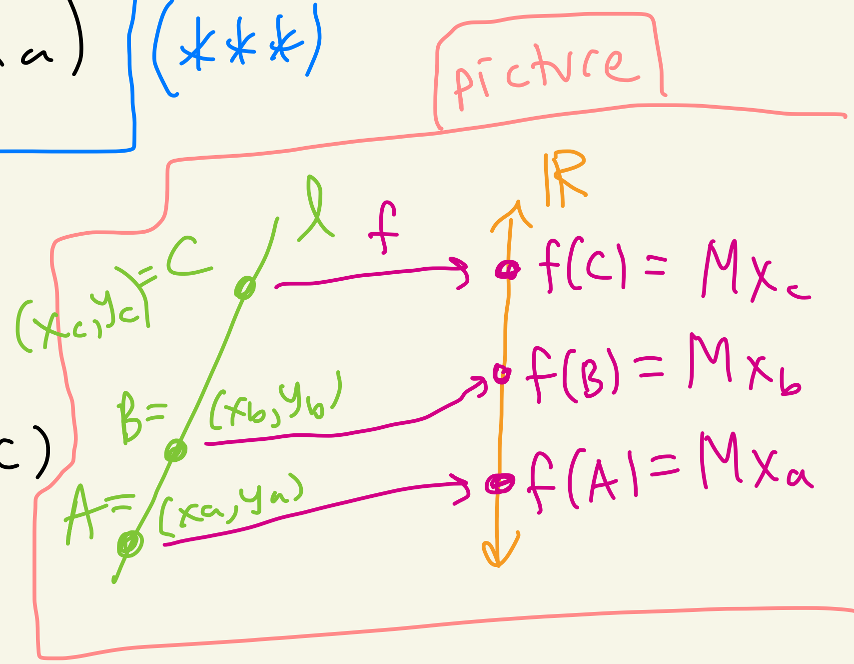
Cancelling M and subtracting x_a gives

$$\boxed{x_b - x_a = t(x_c - x_a)} \quad (***)$$

Since $A-B-C$ we know that either

(i) $f(A) < f(B) < f(C)$

OR (ii) $f(C) < f(B) < f(A)$



That is, either

(i) $Mx_a < Mx_b < Mx_c$

or (ii) $Mx_c < Mx_b < Mx_a$

Since $M > 0$ this becomes either

(i) $x_a < x_b < x_c$

or (ii) $x_c < x_b < x_a$

<u>case (i)</u>: Suppose we have (i).

Then $x_b - x_a > 0$ and $x_c - x_a > 0$.

Thus, from (***) $\underbrace{x_b - x_a}_{>0} = t\underbrace{(x_c - x_a)}_{>0}$

we must have $t > 0$.

We can't have $t \geq 1$ because if we did then we would get

$x_b - x_a = t(x_c - x_a) \geq x_c - x_a$

which would give $x_b \geq x_c$ which isn't true by (i).

Thus, $0 < t < 1$. So we are done with this case.

<u>case (ii)</u>: Suppose we have case (ii)

Then, $x_a - x_c > 0$ and $x_a - x_b > 0$.

So, $x_c - x_a < 0$ and $x_b - x_a < 0$.

Thus, from (***)

$$\underbrace{x_b - x_a}_{<0} = t\underbrace{(x_c - x_a)}_{<0},$$

we must have $t > 0$.

Let's show $t < 1$.

Suppose $t \geq 1$.

Then, $\frac{1}{t} \leq 1$ and $\frac{1}{t}(x_b - x_a) = (x_c - x_a)$

$\times(-1)$

So, $\frac{1}{t}(x_a - x_b) = x_a - x_c$.

Then, $x_a - x_c = \underbrace{\frac{1}{t}}_{\leq 1}(x_a - x_b) \leq x_a - x_b$

So, $-x_c \leq -x_b$.

And then $x_c \geq x_b$.

This contradicts $x_c < x_b$ from case (ii)

Thus, $t < 1$ and so $0 < t < 1$ and we are done with this case.

This concludes the proof of case 2.

Thus we have proven ($\Leftarrow$) since there are only two cases.

$(\Leftarrow)$ Let A, B, C be distinct points where $B = A + t(C-A)$ with $0 < t < 1$.

This implies that $B \in L_{AC} = \overleftrightarrow{AC}$.

So, A, B, C are collinear.

Let $\ell = L_{AC}$.

<u>Goal</u>: We must show that $A-B-C$.

Let $A = (x_a, y_a)$, $B = (x_b, y_b)$, and $C = (x_c, y_c)$.

Then $B = A + t(C-A)$ becomes

$$(\Delta) \quad \underbrace{(x_b, y_b)}_{B} = \underbrace{(x_a + t x_c - t x_a, y_a + t y_c - t y_a)}_{A + t(C-A)}$$

We now break the proof into two cases: when ℓ is vertical and when ℓ is non-vertical.

$\boxed{\text{case 1:}}$ Suppose $\ell = L_d$ is a vertical line

Recall that the standard ruler on ℓ is
$f: \ell \to \mathbb{R}$ given by $f(x,y) = y$.

Apply f to (Δ) above we get

$$y_b = y_a + t y_c - t y_a.$$

So, $\boxed{(y_b - y_a) = t(y_c - y_a)}$ $(\Delta\Delta)$ where $0 < t < 1$.

Since $A \neq B$ and both points lie on the vertical line ℓ we know $y_b \neq y_a$.

Thus, either $y_b - y_a < 0$ or $y_b - y_a > 0$.

$\boxed{\text{case (i)}}$ Suppose $y_b - y_a < 0$

Then from $(\Delta\Delta)$ since we have

$$\underbrace{(y_b - y_a)}_{<0} = \underbrace{t}_{>0}(y_c - y_a)$$

we must have $y_c - y_a < 0$.

Thus, since $y_b - y_a < 0$ and $y_c - y_a < 0$
we have $y_b < y_a$ and $y_c < y_a$.

We want to show that $y_c < y_b$.

Suppose instead that $y_b \leq y_c$.

Since $0 < t < 1$ we know $1 < \frac{1}{t}$.

So, then $\frac{1}{t}(y_b - y_a) = y_c - y_a$ from above
and we get $\frac{1}{t}(y_a - y_b) = y_a - y_c$. (by mult. by -1)

Then,

$$y_a - y_c = \frac{1}{t}(y_a - y_b) \overset{\frac{1}{t} > 1}{>} y_a - y_b \geq y_a - y_c$$

$y_b \leq y_c$ implies $-y_b \geq -y_c$

But then $y_a - y_c > y_a - y_c$, which is a contradiction.

Thus we must have $y_c < y_b$.

Summarizing the above we get $y_c < y_b < y_a$.

Thus, $f(C) < f(B) < f(A)$ and so $C-B-A$.

Since $C-B-A$ we have $A-B-C$.

$\boxed{\text{Case (ii):}}$ Suppose $y_b - y_a > 0$.

Then from $(\Delta\Delta)$ since we have

$$\underbrace{(y_b - y_a)}_{>0} = \underbrace{t}_{>0}(y_c - y_a)$$

we must have $y_c - y_a > 0$

Thus, $y_b - y_a > 0$ and $y_c - y_a > 0$.

So, $y_b > y_a$ and $y_c > y_a$

We want to show that $y_c > y_b$.

Suppose otherwise that $y_c \leq y_b$.

since $y_c - y_a \leq y_b - y_a$ and $t > 0$

Then,

$$y_b - y_a = t(y_c - y_a) \leq t(y_b - y_a)$$
$$< (y_b - y_a)$$

since $t < 1$

This gives $y_b - y_a < y_b - y_a$ which is a contradiction.

Thus, $y_c > y_b$.

So we get $y_a < y_b < y_c$.

So, $f(A) < f(B) < f(C)$

Thus $A - B - C$. This concludes case 1.

$\boxed{\text{case 2:}}$ Suppose $\ell = L_{c,m}$ is a non-vertical line.

Let $f : \ell \to \mathbb{R}$ be the standard ruler
where $f(x,y) = Mx$ where $M = \sqrt{1+m^2} > 0$.

Recall that (Δ) says that

$$(x_b, y_b) = (x_a + t x_c - t x_a, y_a + t y_c - t y_a)$$

Applying f to (Δ) gives

$$M x_b = M(x_a + t x_c - t x_a).$$

Cancelling by M and subtracting x_a gives

$$\boxed{x_b - x_a = t(x_c - x_a)}. \quad (\Delta\Delta\Delta)$$

Since A and B are distinct and they lie on a non-vertical line we know $x_b \neq x_a$ and so $x_b - x_a \neq 0$.

We break the proof into two cases:
$x_b - x_a < 0$ and $x_b - x_a > 0$.

$\boxed{\text{case (i)}}$ Suppose $x_b - x_a < 0$

Then from $(\Delta\Delta\Delta)$ since we have

$$\underbrace{(x_b - x_a)}_{<0} = \underbrace{t}_{>0}(x_c - x_a)$$

we must have $x_c - x_a < 0$.

Thus, since $x_b - x_a < 0$ and $x_c - x_a < 0$

we have $x_b < x_a$ and $x_c < x_a$.

We want to show that $x_c < x_b$.

Suppose instead that $x_b \leq x_c$.

Since $0 < t < 1$ we know $1 < \frac{1}{t}$.

So, then $\frac{1}{t}(x_b - x_a) = x_c - x_a$ from above

and we get $\frac{1}{t}(x_a - x_b) = x_a - x_c$. (by mult. by -1)

Then,

$$x_a - x_c = \frac{1}{t}(x_a - x_b) \overset{\frac{1}{t}>1}{>} x_a - x_b \geq x_a - x_c$$

$x_b \leq x_c$ implies $-x_b \geq -x_c$

But then $x_a - x_c > x_a - x_c$, which is a contradiction.

Thus we must have $x_c < x_b$.

Summarizing the above we get $x_c < x_b < x_a$.

Since $M = \sqrt{1+m^2} > 0$ we get $Mx_c < Mx_b < Mx_a$.

Thus, $f(C) < f(B) < f(A)$ and so $C - B - A$.

Since $C - B - A$ we have $A - B - C$.

$\boxed{\text{Case (ii):}}$ Suppose $x_b - x_a > 0$.

Then from $(\Delta\Delta\Delta)$ since we have

$$\underbrace{(x_b - x_a)}_{>0} = \underbrace{t}_{>0}(x_c - x_a)$$

we must have $x_c - x_a > 0$

Thus, $x_b - x_a > 0$ and $x_c - x_a > 0$.

So, $x_b > x_a$ and $x_c > x_a$

We want to show that $x_c > x_b$.

Suppose otherwise that $x_c \le x_b$.

since $x_c - x_a \le x_b - x_a$ and $t > 0$

Then,

$$x_b - x_a = t(x_c - x_a) \le t(x_b - x_a)$$

$$< (x_b - x_a)$$

since $t < 1$

This gives $x_b - x_a < x_b - x_a$ which is a contradiction.

Thus, $x_c > x_b$.

So we get $x_a < x_b < x_c$.

Since $M = \sqrt{1+m^2} > 0$ this gives $Mx_a < Mx_b < Mx_c$

So, $f(A) < f(B) < f(C)$

Thus $A - B - C$.

This concludes case 2.

Thus, by cases 1 and 2 we always get $A-B-C$.

So we have proven $(\Leftarrow)$. $\blacksquare$